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Absolute Continuity of the Invariant Measures for Some Stochastic PDEs

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We consider a free system and an interacting systems having invariant measures μ and ν respectively. Under suitable assumptions we prove an explicit formula relating ν with μ and implying the absolute continuity of ν with respect to μ. We apply our result to a reaction-diffusion equation and to the Burgers equation.

KEY WORDS: Differential stochastic equations; Ornstein–Uhlenbeck process; invariant measure.

1. INTRODUCTION

We are here concerned with a stochastic differential equation in a Hilbert space H of the following form:

$$dX = (AX + F(X))\, dt + \sqrt{C}\, dW(t), \qquad X(0) = x \in H, \tag{1.1}$$

where $A: D(A) \subset H \to H$ is linear, $C \in L(H)$ is symmetric and nonnegative and $F: D(F) \subset H \to H$ is nonlinear. Moreover $W(t)$ is a cylindrical Wiener process in some probability space $(\Omega, \mathscr{F}, \mathbb{P})$ on H.

Many partial differential equations perturbed by a white noise may be written in this form. This type of model arises in many physical situations. For instance, the stochastic Burgers equation may be considered as a simple model to describe turbulence phenomena.[7, 8, 16, 17] It can also be used in the context of the dynamic of interfaces.[18] It has the form

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$$\begin{cases} \dfrac{\partial}{\partial t} X(t,\xi) = \dfrac{\partial^2}{\partial \xi^2} X(t,\xi) + \dfrac{1}{2}\dfrac{\partial}{\partial \xi}(X(t,\xi)^2) + \dot{\eta}(t,\xi), & t>0, \quad \xi \in (0,1), \\ X(t,\xi)=0, \qquad t>0, \quad \xi = 0,1, \\ X(0,\xi) = x(\xi), \qquad \xi \in (0,1). \end{cases} \tag{1.2}$$

The unknown is X, whose meaning depends on the physical context. It is a random variable which depends on a space variable $\xi \in (0, 1)$ and on the time $t \geqslant 0$. We consider Dirichlet boundary conditions but we could also study other boundary conditions (periodic, Neumann,...). The term $\dot{\eta}$ represents a noise, we consider the case when it is white in time and white or correlated in space.

Equation (1.2) can be written in the form (1.1) if we set[3] $H = L^2(0, 1)$, the space of square integrable functions. The unknown is then considered as a function of the time (and on the random parameter) with values in the Hilbert space H. We also set

$$Ax(\xi) = \frac{\partial^2}{\partial \xi^2} x(\xi), \qquad F(x) = \frac{1}{2}\frac{\partial}{\partial \xi}(x^2).$$

Then, we take

$$\eta = \sqrt{C}\, W$$

where C describes the space correlation of the noise. This equations has been extensively studied and, if C is a bounded operator, it is known that there exists a unique global solution which is a continuous process with values in H (see ref. 11). It is convenient to emphasize the dependence of the solution with respect to the initial data x so that we denote the solution by $X(t, x)$.

It is often important to understand the long time behaviour of the system described by these equations. In many circumstances there exists an invariant measure ν which describes this behaviour. For, instance if the system is strongly mixing, we know that for every continuous and bounded functional φ defined on H we have, for any x,

$$\lim_{t\to\infty} \varphi(X(t,x)) = \int_H \varphi(y)\, \nu(dy).$$

The measure ν is also often called an equilibrium measure. The existence of an invariant measure for the Burgers equation was proved in ref. 11. In

[3] These mathematical object will be rigourously defined in Section 4.2 later.

ref. 12, it was also proved that this invariant measure is unique and that the strong mixing property holds. This latter result has been proved under the assumptions that C is invertible, in other words when the noise is also white in space. However, more recently, more refined techniques have been developped in the more difficult case of the Navier–Stokes to prove this result under much weaker assumptions.[4, 15, 19]

It is then an important problem to understand the structure of this measure ν. In particular, we want to know if it can be described in terms of a density ρ. In the finite dimensional case,[4] it is natural to try to write $\nu(dx) = \rho(x)\,dx$, where dx is the Lebesgue measure. This problem has been solved in a very general way thanks to Malliavin calculus (see, for instance, ref. 20).

However, it is well known that in the infinite dimensional case considered here, the Lebesgue measure cannot be defined and no such reference measure exists. Moreover, up to now, the Malliavin calculus has not been generalized in a satisfactory way to this context.

In several situations problem (1.1) describes the evolution of an *interacting* stochastic system, the corresponding *free* system being described by the linear equation

$$dZ = AZdt + \sqrt{C}\, dW(t), \qquad Z(0) = x \in H, \tag{1.3}$$

where $Z(t, x)$ is an Ornstein–Uhlenbeck process. This system also has an invariant measure μ. It is Gaussian and, if A and C commute, is formally given by

$$\mu(dx) = \frac{1}{\beta} \exp\left(-\frac{1}{2} |(-CA)^{1/2}\, x|^2 \right) dx$$

where $|\cdot|$ is the norm in H and β a normalizing factor.

Then, we can try to replace the finite dimensional Lebesgue measure by μ and to prove that ν is absolutely continuous with respect to μ. This implies the existence of a density ρ which satisfies

$$\nu(dx) = \rho(x)\, \mu(dx).$$

This problem has been considered mainly when (1.1) describes a reversible system, see, e.g., refs. 2, 23, and 24. In this case the explicit expressions of ν and μ are often available, and so the answer is not difficult

[4] This is the case when stochastic differential equations are considered instead of a stochastic partial differential equation so that $H = \mathbb{R}^d$, a finite dimensional space.

in general. For instance if C is the identity operator and $F = DU$ is the differential of a potential U, then the system is gradient and $\rho(x) = \exp U(x)$.

When the system is not reversible the situation is more involved. Under suitable assumptions, requiring that C has a bounded inverse and

$$\int_H |F(x)|^2 \, \nu(dx) < +\infty, \tag{1.4}$$

one can prove that ν is abolutely continuous with respect to μ by the method presented in ref. 3. However, (1.4) is not fulfilled in some interesting cases such as the Burgers equation. We shall present here a general approach which applies to several cases as: reaction-diffusion equations, Burgers equation, Navier–Stokes equations.

Let us explain the main idea of our method. Assume that we are able to solve Eqs. (1.1) and (1.3) and denote by P_t and R_t the corresponding transition semigroups:

$$P_t\varphi(x) = \mathbb{E}[\varphi(X(t, x))], \qquad \varphi \in B_b(H)$$

and

$$R_t\varphi(x) = \mathbb{E}[\varphi(Z(t, x))], \qquad \varphi \in B_b(H),$$

where $B_b(H)$ is the Banach space of all Borel and bounded mappings $\varphi: H \to \mathbb{R}$, endowed with the sup norm. Let us define the infinitesimal generators of N and L through the resolvent formulae, see ref. 5 and Section 2,

$$(\lambda - N)^{-1} f(x) = \int_0^\infty e^{-\lambda t} P_t f(x) \, dt, \qquad \lambda > 0, \quad f \in B_b(H),$$

and

$$(\lambda - L)^{-1} f(x) = \int_0^\infty e^{-\lambda t} R_t f(x) \, dt, \qquad \lambda > 0, \quad f \in B_b(H).$$

Formally, we have $P_t\varphi = e^{Nt}\varphi$, $R_t\varphi = e^{Lt}\varphi$ and the invariant measure satisfy

$$\int_H N\varphi(x) \, \nu(dx) = 0, \qquad \int_H L\varphi(x) \, \mu(dx) = 0,$$

for sufficiently smooth φ.

Our main tool is the following identity

$$(\lambda - L)^{-1} f = (\lambda - N)^{-1} f - (\lambda - N)^{-1} [\langle F, D(\lambda - L)^{-1} f \rangle], \qquad f \in B_b(H). \tag{1.5}$$

We shall show that, assuming that the semigroup R_t is strong Feller, the function $D(\lambda - L)^{-1} f$ is well defined and continuous for any $f \in B_b(H)$; moreover it can be extended up to $\lambda = 0$. Assume in addition that

$$\lim_{\lambda \to 0} \lambda(\lambda - N)^{-1} \varphi(x) = \int_H \varphi(y)\, \nu(dy), \qquad \text{for } \nu\text{-almost all} \quad x \in H, \tag{1.6}$$

and

$$\lim_{\lambda \to 0} \lambda(\lambda - L)^{-1} \varphi(x) = \int_H \varphi(y)\, \mu(dy), \qquad \text{for all} \quad x \in H. \tag{1.7}$$

Then we will show, letting $\lambda \to 0$ in (1.5) that

$$\int_H f\, d\mu = \int_H f\, d\nu + \int_H \langle F, DL^{-1} f \rangle\, d\nu, \qquad f \in B_b(H). \tag{1.8}$$

As already mentionned, we will see that the term $DL^{-1}f$ can be defined rigourously. From (1.8) we can show easily that ν is absolutely continuous with respect to μ, see Section 3 later, implying the existence of a density. Notice that (1.7) is in general very easy to check, see Section 2, whereas (1.6) requires that the measure ν is strongly mixing.

In Section 4, we apply our results to a reaction-difusion equation and to the Burgers equation with correlated noise. This corresponds to a noninvertible C and we can treat the case of a nonsymmetric Ornstein–Uhlenbeck process, so that our result does not does not follow from ref. 3.

Finally, our result can be applied to the two dimensional stochastic Navier–Stokes equation as will be shown in a forthcoming article.

2. THE FREE SYSTEM

Concerning the linear operators A and C we shall assume:

Hypothesis 2.1.

(i) A is the infinitesimal generator of an analytic semigroup e^{tA} in H. There exists $M, \omega > 0$ such that $\|e^{tA}\| \leqslant Me^{-\omega t}$, for all $t \geqslant 0$.

(ii) $C: H \to H$ is bounded and nonnegative and the linear operator Q defined as

$$Qx = \int_0^{+\infty} e^{tA} C e^{tA^*} x \, dt, \qquad x \in H,$$

is of trace class.[5] We shall denote by $\mu = N_Q$ the Gaussian measure with mean 0 and covariance operator Q.

(iii) For all $t > 0$ we have $e^{tA}(H) \subset Q_t^{1/2}(H)$, where

$$Q_t x = \int_0^t e^{sA} C e^{sA^*} x \, dt, \qquad x \in H.$$

(iv) Setting $\Lambda_t = Q_t^{-1/2} e^{tA}$, the function $\|\Lambda_t\|$ is Laplace transformable with Laplace transform:

$$\gamma(\lambda) := \int_0^{+\infty} e^{-\lambda t} \, \|\Lambda_t\| \, ds,$$

defined in $(-\omega, +\infty)$.

Let us give some comments about Hypothesis 2.1. Under assumptions (i) and (ii) we can consider the Ornstein–Uhlenbeck semigroup R_t

$$R_t \varphi(x) = \int_H \varphi(e^{tA} x + y) \, N_{Q_t}(dy), \qquad \varphi \in B_b(H), \tag{2.1}$$

where N_{Q_t} is the Gaussian measure in H with mean 0 and covariance operator Q_t. Moreover μ is the unique invariant mesure of R_t and we have

$$\lim_{t \to +\infty} R_t \varphi(x) = \int_H \varphi(y) \, N_Q(dy), \qquad \text{for all} \quad \varphi \in B_b(H), \quad x \in H. \tag{2.2}$$

If, in addition, Hypothesis 2.1(iii) is fulfilled, then R_t is strong Feller[6] and the following result holds, see ref. 13.

Proposition 2.2. Assume that Hypothesis 2.1(i)–(iii) holds. Then for all $t > 0$ and $\varphi \in B_b(H)$ we have $R_t \varphi \in C_b^1(H)$,[7] and

$$\langle DR_t \varphi, h \rangle = \int_H \langle \Lambda_t h, Q_t^{-1/2} y \rangle \, \varphi(e^{tA} x + y) \, N_{Q_t}(dy) \tag{2.3}$$

[5] A^* is the adjoint of A.

[6] R_t is strong Feller if and and only if for all $t > 0$ and Borel and bounded φ, $R_t \varphi$ is continuous.

[7] $C_b(H)$ is the Banach space of all uniformly continuous and bounded mappings $\varphi: H \to \mathbb{R}$, endowed with the norm $\|\varphi\|_0 = \sup_{x \in H} |\varphi(x)|$. For $k \in \mathbb{N}$, $C_b^k(H)$ is defined in the usual way.

for all $h \in H$. Moreover

$$\|DR_t\varphi\|_0 \leqslant \|\Lambda_t\| \, \|\varphi\|_0, \qquad t > 0. \tag{2.4}$$

Hypothesis 2.1(iv) is unusual but it will play an essential rôle in the following. Let us give an example where it is fulfilled.

Example 2.3. Let A satisfy Hypothesis 2.1(i) and let C be such that $C(-A)^\delta$ is a bounded operaator for some $\delta \in [0, 1)$. Then by ref. 14, Section 13.1, we have

$$\|\Lambda_t\| \leqslant K_1 t^{-\frac{1+\delta}{2}}, \qquad t \in [0, 1],$$

for a constant K_1. Moreover,[22] we remark that $\Sigma_t := \Lambda_t \Lambda_t^*$ satisfies a Riccati equation associated to a control problem (see ref. 21) and for $t \geqslant t_0 > 0$, we have

$$\langle \Sigma_t x, x\rangle = \inf \left\{ \int_{t_0}^t |u(s)|^2 \, ds + \langle \Sigma_{t_0} y^{x,u}(t), y^{x,u}(t)\rangle \right\},$$

where $y^{x,u}$ is the solution of

$$y' = Ay + \sqrt{C}\, u, \qquad y(t_0) = x.$$

Therefore, choosing $u = 0$ we obtain

$$\begin{aligned}\langle \Sigma_t x, x\rangle &\leqslant \langle \Sigma_{t_0} y^{x,0}(t), y^{x,0}(t)\rangle \\ &\leqslant \|e^{tA^*} \Sigma_{t_0} e^{tA}\| \, |x|^2 \leqslant K_2 e^{-2\omega t} \, |x|^2,\end{aligned}$$

for a constant K_2. We deduce

$$\|\Lambda_t\| \leqslant K_2^{1/2} e^{-\omega t}$$

so that Hypothesis 2.1(iv) holds. It is standard that Hypothesis 2.1(i) holds for any $\delta \in [0, 1)$ and that Hypothesis 2.1(ii) holds provided

$$\mathrm{Tr}[(-A)^{-(\delta+1)}] < \infty.$$

We recall that R_t is not a strongly continuous semigroup neither in $C_b(H)$ nor in $B_b(H)$ when $A \neq 0$. However we can define its infinitesimal generator L, through its resolvent as in ref. 5. In fact, setting

$$F(\lambda)\, f(x) = \int_0^\infty e^{-\lambda t} R_t f(x) \, dt, \qquad \lambda > 0, \quad f \in B_b(H),$$

it is not difficult to show that $F(\lambda)$ maps $B_b(H)$ into $B_b(H)$ and it is one-to-one. Consequently, there exists a unique linear closed operator L in $B_b(H)$ such that

$$(\lambda - L)^{-1} f(x) = \int_0^\infty e^{-\lambda t} R_t f(x)\, dt, \qquad \lambda > 0, \quad f \in B_b(H).$$

The following result is a corollary of Proposition 2.2 via Laplace transform.

Proposition 2.4. Assume that Hypothesis 2.1 holds. Let $f \in B_b(H)$ and $\lambda > 0$. Then $\varphi = (\lambda - L)^{-1} f \in C_b^1(H)$ and

$$\|D\varphi\|_0 \leqslant \gamma(\lambda)\, \|f\|_0. \tag{2.5}$$

Consequently

$$D(L) \subset C_b^1(H), \tag{2.6}$$

with continuous embedding.

We end this section by proving some results about the behaviour of $(\lambda - L)^{-1}$ as $\lambda \to 0$; we shall see that, whereas $(\lambda - L)^{-1}$ is singular, $D(\lambda - L)^{-1} f$ has a limit when $\lambda \to 0$.

Proposition 2.5. Assume that Hypothesis 2.1 holds. Then for any $f \in B_b(H)$ there exist the limits

$$\lim_{\lambda \to 0} \lambda(\lambda - L)^{-1} f(x) = \int_H f\, d\mu, \qquad \text{for all} \quad x \in H, \tag{2.7}$$

$$\lim_{\lambda \to 0} D(\lambda - L)^{-1} f(x) = \int_0^{+\infty} DR_t f(x)\, dt$$

$$:= -DL^{-1} f(x), \qquad \text{for all} \quad x \in H. \tag{2.8}$$

Moreover $DL^{-1} f \in C_b(H)$.

Proof. For any $f \in B_b(H)$ we have

$$\lambda(\lambda - L)^{-1} f(x) = \int_0^{+\infty} e^{-\tau} R_{\tau/\lambda} f(x)\, d\tau,$$

and so (2.7) follows from (2.2). Let us prove (2.8). Since $\gamma(\lambda)$ is defined in $(-\omega, +\infty)$, we have

$$\int_0^{+\infty} \|\Lambda_t\| \, dt < +\infty.$$

Using (2.4) this implies that $|DR_t\varphi(x)|$ is summable in $[0, +\infty)$ and that (2.8) holds. ∎

3. THE INTERACTING SYSTEM

Here we assume, besides Hypothesis 2.1, that

Hypothesis 3.1.

(i) The differential stochastic equation

$$dX = (AX + F(X))\, dt + \sqrt{C}\, dW(t), \qquad X(0) = x \in H, \tag{3.1}$$

has a unique mild solution $X(t, x)$. That is there exists a unique adapted stochastic process $X(\cdot, x)$ such that

$$X(t, x) = e^{tA}x + \int_0^t e^{(t-s)A} F(X(s, x))\, ds + \int_0^t e^{(t-s)A} \sqrt{C}\, dW(s), \qquad \mathbb{P}\text{-a.s.}$$

We denote by P_t the corresponding transition semigroup

$$P_t\varphi(x) = \mathbb{E}[\varphi(X(t, x))], \qquad \varphi \in B_b(H).$$

(ii) The semigroup P_t is Feller and has an invariant measure ν.

We denote by N its infinitesimal generator, defined by

$$(\lambda - N)^{-1}\, \varphi(x) = \int_0^{\infty} e^{-\lambda t} P_t\varphi(x)\, dt, \qquad \lambda > 0, \quad \varphi \in B_b(H).$$

It is standard that P_t can be extended to a contraction semigroup on $L^2(H, \nu)$.

(iii) The measure ν is strongly mixing

$$\lim_{t \to +\infty} P_t\varphi(x) = \int_H \varphi(y)\, \nu(dy), \qquad \nu\text{-a.s. in} \quad H\varphi \in L^2(H, \nu). \tag{3.2}$$

These are the key assumptions. We now set technical hypotheses which are easy to check in the applications even if their proofs may involve tedious computations.

(iv) There exists a sequence $\{F_n\}$ of Lipschitz continuous mappings from H into H such that

$$F_n(x) \to F(x), \qquad \nu\text{-a.s. in } H$$

and a function $g \in L^2(H, \nu)$ such that

$$|F_n(x)| \leqslant g(x), \qquad x \in H.$$

It is well known that, under Hypothesis 3.1(iv), problem

$$dX = (AX + F_n(X))\,dt + \sqrt{C}\,dW(t), \qquad X(0) = x \in H, \tag{3.3}$$

has a unique mild solution $X_n(t, x)$. Let us denote by P_t^n the corresponding transition semigroup

$$P_t^n \varphi(x) = \mathbb{E}[\varphi(X_n(t, x))], \qquad \varphi \in B_b(H),$$

and by N_n its infinitesimal generator, defined by

$$(\lambda - N_n)^{-1}\,\varphi(x) = \int_0^\infty e^{-\lambda t} P_t^n \varphi(x)\,dt, \qquad \lambda > 0, \quad \varphi \in B_b(H).$$

(v) For all $t > 0$ and ν almost every $x \in H$

$$X_n(t, x) \to X(t, x), \qquad \mathbb{P}\text{-a.s. in } H$$

(vi) For all $\lambda > 0$ and ν almost every $x \in H$.

$$\int_0^\infty e^{-\lambda t}\,|F_n(X_n(t, x))) - F_n(X(t, x))|\,dt \to 0 \qquad \text{as} \quad n \to \infty.$$

Clearly, (v) implies that

$$(\lambda - N_n)^{-1}\,\varphi(x) \to (\lambda - N)^{-1}\,\varphi(x), \qquad \text{for all} \quad x \in H, \quad \varphi \in C_b(H). \tag{3.4}$$

Also, since P_t is Feller, $C_b(H)$ is an invariant for N in $L^2(H, \nu)$ and we have

$$(\lambda - N)^{-1}\,\varphi(x) = \int_0^\infty e^{-\lambda t} P_t \varphi(x)\,dt, \qquad \lambda > 0, \quad \varphi \in L^2(H, \nu).$$

Writing

$$\lambda(\lambda - N)^{-1}\,\varphi = \int_0^\infty e^{-\tau} P_{\tau/\lambda}\varphi(x)\,dt,$$

we deduce from (iii) that

$$\lambda(\lambda - N)^{-1}\,\varphi \to \int_H \varphi(y)\,\nu(dy), \qquad \text{in } L^2(H, \nu), \quad \varphi \in L^2(H, \nu). \tag{3.5}$$

Remark 3.2. It is easy to check that, since F_n is Lipschitz continuous, there exists a positive constant M_n such that

$$\mathbb{E}(|X_n(t, x)|) \leqslant M_n(1 + |x|), \qquad x \in H.$$

It follows that the transition semigroup P_t^n can be extended to the space $B_{b,1}(H)$ of all Borel functions $\varphi: H \to \mathbb{R}$ such that $\varphi(1 + |x|)^{-1}$ is bounded. In the same way one can extend $(\lambda - N_n)^{-1}$ to $B_{b,1}(H)$.

We now prove a basic identity for the regularized equation (3.3).

Lemma 3.3. Assume that Hypotheses 2.1 and 3.1 hold. Then for any $\lambda > 0$, $n \in \mathbb{N}$ and any $f \in B_b(H)$ the following identity holds

$$(\lambda - L)^{-1} f = (\lambda - N_n)^{-1} f - (\lambda - N_n)^{-1}\,[\langle F_n, D(\lambda - L)^{-1} f\rangle]. \tag{3.6}$$

Proof. Set, for any $\varepsilon > 0$,

$$F_{n,\varepsilon}(x) = \frac{F_n(x)}{1 + \varepsilon\,|x|}, \qquad x \in H.$$

Then $F_{n,\varepsilon}$ are Lipschitz continuous, uniformly in ε, and bounded. We denote by $X_{n,\varepsilon}(t, x)$ the mild solution of the differential stochastic equation

$$dX = (AX + F_{n,\varepsilon}(X))\,dt + \sqrt{C}\,dW_t, \qquad X(0) = x,$$

by $P_t^{n,\varepsilon}$ the corresponding transition semigroup

$$P_t^{n,\varepsilon}\varphi(x) = \mathbb{E}[\varphi(X_{n,\varepsilon}(t, x))], \qquad \varphi \in B_b(H),$$

and by $N_{n,\varepsilon}$ its infinitesimal generator defined as before. Now, let $\lambda > 0$ and $f \in B_b(H)$. Consider the following equation

$$\lambda\varphi_{n,\varepsilon} - L\varphi_{n,\varepsilon} - \langle F_{n,\varepsilon}, D\varphi_{n,\varepsilon}\rangle = f. \tag{3.7}$$

Notice that Eq. (3.7) is meaningful in view of (2.6). Setting $\lambda\varphi_{n,\varepsilon} - L\varphi_{n,\varepsilon} = \psi_{n,\varepsilon}$, (3.7) becomes

$$\psi_{n,\varepsilon} - T_\lambda^{n,\varepsilon}\psi_{n,\varepsilon} = f, \tag{3.8}$$

where

$$T_\lambda^{n,\varepsilon}\psi = \langle F_{n,\varepsilon}, DR(\lambda - L)^{-1}\,\psi\rangle, \qquad \psi \in B_b(H).$$

But, in view of Proposition 2.4, we have

$$\|T_\lambda^{n,\varepsilon}\psi\|_0 \leqslant \gamma(\lambda)\,\|F_{n,\varepsilon}\|_0\,\|\psi\|_0, \qquad \psi \in B_b(H).$$

Since $\lim_{\lambda \to +\infty} \gamma(\lambda) = 0$, there exists a positive number $\lambda_{n,\varepsilon}$ such that, if $\lambda > \lambda_{n,\varepsilon}$, Eq. (3.8) can be uniquely solved by a standard fixed point argument. In conclusion, $(\lambda_{n,\varepsilon}, \infty)$ belongs to the resolvent set of $N_{n,\varepsilon}$ and we have

$$(\lambda - N_{n,\varepsilon})^{-1} = (\lambda - L)^{-1}\,(1 - T_\lambda^{n,\varepsilon})^{-1}, \qquad \text{for} \quad \lambda > \lambda_{n,\varepsilon}.$$

It follows that, for $\lambda > \lambda_{n,\varepsilon}$,

$$(\lambda - L)^{-1}\,f = (\lambda - N_{n,\varepsilon})^{-1}\,f - (\lambda - N_{n,\varepsilon})^{-1}\,[\langle F_{n,\varepsilon}, D(\lambda - L)^{-1}\,f\rangle]. \tag{3.9}$$

Now, by analytic continuation (3.9) holds for any $\lambda > 0$. Finally the conclusion follows by letting ε tend to 0 taking into account Remark 3.2. ▮

Theorem 3.4. Assume that Hypotheses 2.1 and 3.1 hold. Let μ and ν be the invariant measures of R_t and P_t respectively. Then for any $f \in B_b(H)$ we have

$$\int_H f\,d\mu = \int_H f\,d\nu + \int_H \langle F, DL^{-1}f\rangle\,d\nu. \tag{3.10}$$

Moreover ν is absolutely continuous with respect to μ.

Proof. By Lemma 3.3 we have, for any $n \in N$, $\lambda > 0$,

$$(\lambda - L)^{-1}\,f = (\lambda - N_n)^{-1}\,f - (\lambda - N_n)^{-1}\,[\langle F_n, D(\lambda - L)^{-1}\,f\rangle]. \tag{3.11}$$

Let us assume for the moment that $f \in C_b(H)$. By (3.4) we know that

$$(\lambda - N_n)^{-1}\,f(x) \to (\lambda - N)^{-1}\,f(x), \qquad \text{for all} \quad x \in H.$$

The second term is more delicate to treat. We first write it as follows

$$(\lambda-N_n)^{-1}[\langle F_n, D(\lambda-L)^{-1} f\rangle]$$
$$=(\lambda-N)^{-1}[\langle F, D(\lambda-L)^{-1} f\rangle]+A_n+B_n,$$

with

$$A_n=(\lambda-N)^{-1}[\langle F_n-F, D(\lambda-L)^{-1} f\rangle]$$

and

$$B_n=((\lambda-N_n)^{-1}-(\lambda-N)^{-1})[\langle F_n, D(\lambda-L)^{-1} f\rangle].$$

Using dissipativity of N in $L^2(H,\nu)$, Proposition 2.4 and Assumption (3.1)(iv), we have

$$\|A_n\|_{L^2(H,\nu)} \leqslant \frac{1}{\lambda}\gamma(\lambda)\,\|f\|_0\,\|F_n-F\|_{L^2(H,\nu)} \to 0,$$

when $n\to\infty$.

We split again B_n as follows

$$B_n=B_n^1+B_n^2,$$

$$B_n^1(x)=\int_0^\infty e^{-\lambda t}\,\mathbb{E}\langle F_n(X_n(t,x))-F_n(X(t,x)), f^\lambda(X_n(t,x))\rangle\,dt,$$

$$B_n^2(x)=\int_0^\infty e^{-\lambda t}\,\mathbb{E}\langle F_n(X_n(t,x)), (f^\lambda(X_n(t,x))-f^\lambda(X(t,x)))\rangle\,dt$$

where $f^\lambda=D(\lambda-L)^{-1}f$. By Proposition 2.4 and Assumption (3.1)(vi)

$$|B_n^1(x)|\leqslant \gamma(\lambda)\,\|f\|_0\int_0^\infty e^{-\lambda t}\mathbb{E}\,|F_n(X_n(t,x))-F_n(X(t,x))|\,dt\to 0,$$

for ν–almost every $x\in H$. Moreover, by Assumption (3.1)(iv)

$$\|B_n^2\|_{L^1(H,\nu)}\leqslant \int_0^\infty\int_H \mathbb{E}(g(X(t,x)\,|f^\lambda(X_n(t,x))-f^\lambda(X(t,x))|)\,d\nu\,dt,$$

By Assumption (3.1)(v)

$$g(X(t,x)\,|f^\lambda(X_n(t,x))-f^\lambda(X(t,x))|\to 0,\qquad dt\times d\nu\times d\mathbb{P}\text{-a.e.}$$

Moreover, by Proposition 2.4 and the invariance of ν

$$\int_0^\infty e^{-2\lambda t}\,\mathbb{E}(g(X(t,x)^2\,|f^\lambda(X_n(t,x))-f^\lambda(X(t,x))|^2)\,\nu(dx)\,dt$$
$$\leqslant 4\,\frac{\gamma^2(\lambda)}{2\lambda}\,\|f\|_0^2\,\|g\|_{L^2(H,\nu)}.$$

By uniform integrability, we obtain

$$\|B_n^2\|_{L^1(H,\nu)}\to 0.$$

Gathering these results, we deduce that there exists a subsequence such that

$$(\lambda-N_{n_k})^{-1}\,[F_{n_k}D(\lambda-L)^{-1}\,f](x)\to(\lambda-N)^{-1}\,[FD(\lambda-L)^{-1}\,f](x), \quad (3.12)$$

ν almost surely. We obtain

$$(\lambda-L)^{-1}\,f=(\lambda-N)^{-1}\,f-(\lambda-N)^{-1}\,[\langle F, D(\lambda-L)^{-1}\,f\rangle], \qquad (3.13)$$

for any $f\in C_b(H)$. It is now easy to extend this to any $f\in B_b(H)$ by taking a sequence $f_n\in C_b(H)$ which converges pointwise to f.

Now, multiplying both sides of (3.13) by λ and letting $\lambda\to 0$ yields (3.9) thanks to (2.7) and (3.5).

Let us now prove the absolute continuity of ν with respect to μ. Let $\Gamma\subset H$ be a Borel set such that $\mu(\Gamma)=0$. Then we have

$$R_t\chi_\Gamma(x)=N_{e^{tA}x,\,Q_t}(\Gamma)=0, \qquad \text{for all} \quad t>0 \quad \text{and} \quad x\in H.$$

This follows from the well known fact that, since R_t is strong Feller, the measure $N_{e^{tA}x,\,Q_t}$ is absolutely continuous with respect to μ. Consequently, $D(\lambda-L)^{-1}\,\chi_\Gamma(x)=0$ for all $x\in H$. Thus, by (3.9) it follows that $\nu(\Gamma)=\mu(\Gamma)=0$. ∎

4. APPLICATIONS

4.1. Reaction-Diffusion Equations

Let D be a bounded subset of $\mathbb{R}^d$ with regular boundary ∂D. Let us consider the following problem

$$\begin{cases} dX(t,\xi)=(\Delta_\xi X(t,\xi)+p(X(t,\xi)))\,dt+\sqrt{C}\,dW(t,\xi), & t>0, \quad \xi\in D,\\ X(t,\xi)=0, \qquad t>0, \quad \xi\in\partial D, & \\ X(0,\xi)=x(\xi), \qquad \xi\in D, & \end{cases} \qquad (4.1)$$

where Δ_ξ is the Laplace operator, p is a polynomial of odd degree N having negative leading coefficient, W is space time Brownian sheet, and C is such that $C(-\Delta)^\delta$ is bounded for some $\delta > 0$.[8] Setting

$$H = L^2(D),$$

$$Ax(\xi) = \Delta_\xi x(\xi), \qquad x \in D(A) = H^2(D) \cap H_0^1(D),$$

and

$$F(x)(\xi) = p(x(\xi)), \qquad x \in L^{2N}(D),$$

problem (4.1) becomes equivalent to problem (1.1).

It is well known that A generates an analytic semigroup of negative type in H. As discussed in Example 2.3, Hypothesis 2.1 is fulfilled if $\delta \in [0, 1)$ and

$$\mathrm{Tr}(-\Delta)^{-1-\delta} < +\infty. \tag{4.2}$$

Since the eigenvalues of A behave asymptotically as $k^{2/d}$ when $k \to \infty$, see ref. 1, we find that (4.2) holds provided

$$\sum_{k=1}^{\infty} k^{-2(1+\delta)/d} < +\infty, \tag{4.3}$$

or, equivalently, to $2(1+\delta) > d$.

In conclusion, Hypothesis 2.1 is fulfilled provided $\delta \in [0, 1)$ if $d = 1$, $\delta \in (0, 1)$ if $d = 2$, $\delta \in (1/2, 1)$ if $d = 3$. In this case the free system has a unique invariant measure. If $d > 3$ Hypothesis 2.1 does not hold.

Let us check now Hypothesis 3.1. Concerning (i), we recall that existence and uniqueness of a solution of Eq. (4.1) is well known, see, e.g., refs. 6 and 13. In the monograph of ref. 6, more general situations are considered such as systems of reaction-diffusion equations. Existence of an invariant measure can be found in ref. 13. Also in ref. 6 it was proved that the semigroup P_t is irreducible and strong-Feller. Therefore, in view of the Doob theorem, see, e.g., ref. 13, Assumptions 3.1(ii) and (iii) are fulfilled.

Now we set

$$F_n(x)(\xi) = \frac{p(x(\xi))}{1+\frac{1}{n}\,x(\xi)^{N-1}}, \qquad x \in D.$$

[8] In the definition of C we assume that the Laplacian is supported with Dirichlet boundary conditions.

In order to avoid technical difficulties. we restrict our attention to the case $D=[0,1]^d$, although the result can be extended to more general domains. In that case the invariant mesure ν is supported by $C_b(D)$, the space of all bounded and continuous functions on D. Moreover, for any $k\in\mathbb{N}$, $p\in[1,\infty]$

$$\int_H |x|^k_{L^p(D)}\,\nu(dx)<+\infty.$$

This easily implies Hypothesis 3.1(iv). Also it follows from ref. 6 that, if $x\in C_b(D)$

$$X_n(t,x)\to X(t,x)\qquad \mathbb{P}\text{-a.s.}$$

in $C_b(D)$ and in H. This implies 3.1(v) and

$$F_n(X_n(t,x))-F_n(X(t,x))\to 0\qquad \text{in } H,\quad dt\times\mathbb{P}\text{-a.e.},\tag{4.4}$$

for any $x\in C_b(D)$ and thus ν almost surely. The following lemma involves some computations and it is left to the reader.

Lemma 4.1. For any $p\geqslant 1$ there exists a constant $c(p)$ such that for any $x\in C_b(D)$,

$$\mathbb{E}(|X_n(t,x)|^p_{L^p(D)})\leqslant c(p)(|x|^p_{L^p(D)}+1),\qquad \forall n\in\mathbb{N},\quad t\geqslant 0.$$

Then Lemma 4.1 implies Hypothesis 3.1(vi) by uniform integrability and (4.4).

Therefore, by Theorem 3.4 we find the following result.

Theorem 4.2. Let p be a polynomial of odd degree N, having negative leading cofficient and let C be such that $C(-A)^\delta$ is bounded with $\delta\in[0,1)$ if $d=1$, $\delta\in(0,1)$ if $d=2$ and $\delta\in(1/2,1)$ if $d=3$. Then problem (4.1) has a unique mild solution and there exists a unique invariant measure ν. Moreover if $D=[0,1]^d$, ν is absolutely continuous with respect to the invariant measure μ of the free system.

4.2. Burgers Equation

We are here concerned with the following problem

$$\begin{cases} dX(t,\xi)=(\Delta_\xi X(t,\xi)+\frac{1}{2}D_\xi(X(t,\xi)^2)\,dt+\sqrt{C}\,dW(t,\xi), \\ \qquad\qquad\qquad\qquad t>0,\quad \xi\in(0,1), \\ X(t,\xi)=0, \qquad t>0,\quad \xi=0,1, \\ X(0,\xi)=x(\xi), \qquad \xi\in(0,1), \end{cases}\tag{4.5}$$

where Δ_ξ and W are as before, $H = L^2(0, 1)$, C is such that $C(-A)^\delta$ is bounded with $\delta \in [0, 1)$,

$$Ax(\xi) = \Delta_\xi x(\xi), \qquad x \in D(A) = H^2((0, 1)) \cap H_0^1((0, 1)),$$

and

$$F(x) = \tfrac{1}{2} D_\xi(x^2), \qquad x \in H_0^1(0, 1),$$

problem (4.5) becomes equivalent to problem (1.1). By proceeding as in the previous section, it is easy to see that if $\delta \in [0, 1)$ Hypothesis 2.1 is fulfilled, and that the free system has a unique invariant measure.

Existence of an invariant measure is also proved in ref. 11. Note that in ref. 11 only the case $\delta = 0$ was considered, but this result can be easily extended to the case $\delta > 0$.

In order to satisfy Hypothesis 3.1(iv), we need that $F(x) \in H$ ν–almost surely. This requires that ν is supported by $H_0^1(0, 1)$ which is the case if $\delta > 1/2$. Under that condition, it follows from ref. 9 that P_t is strong Feller. Irreducibility can be shown by a control argument as in ref. 13. Thus Hypothesis 3.1(iii) is fulfilled by the Doob theorem. Finally, Hypotheses 3.1(iv)–(v) are satisfied with

$$F_n(x) = \frac{n}{n + |x|} P_n F(P_n x), \qquad n \in \mathbb{N},$$

where P_n is the projector on the linear span of the first n eigenvectors of A.

This follows from similar arguments as in ref. 10. It is easy to see that

$$|F_n(x)| \leqslant c_1 \, |x|^2_{H_0^1(0, 1)}$$

for a constant c_1 which does not depend on n. Moreover, using the same computation as in ref. 10, Proposition 2.6, we have

$$\mathbb{E}(|X_n(t, x)|^4_{H_0^1(0, 1)}) \leqslant c_2(|x|^4_{H_0^1(0, 1)} + 1),$$

for a constant c_2 independent on t, n, x. It follows that

$$\mathbb{E}(|F_n(X_n(t, x))|^2) \leqslant c_1 c_2(|x|^4_{H_0^1(0, 1)} + 1)$$

and (vi) is obtained by uniform integrability.

Therefore, by Theorem 3.4 we find the following result.

Theorem 4.3. There exists a unique invariant measure ν for problem (4.5). Moreover ν is absolutely continuous with respect to the invariant measure μ of the corresponding free system.

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